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LETTER TO THE EDITOR

A multidimensional superposition principle and wave switching in integrable and nonintegrable soliton models

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Abstract
In the framework of a multidimensional superposition principle a series of computer experiments with integrable and nonintegrable models are carried out with the goal of verifying the existence of switching effect and superposition in soliton–perturbation interactions for a wide class of nonlinear PDEs.

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1. Introduction

One of the reasons for the appearance of a multidimensional superposition principle (MSP) [1, 2] was an attempt to explain an unusual type of kink interactions, namely their switching from one state to another with different wave numbers, described in [3]. In effect, if we do not take into account the mathematical background, the method is an exact version of the well-known collective-variable approach (see, e.g., references in [4]). In this framework, speaking about soliton–perturbation interactions, the soliton's parameters become functions depending on a perturbation, so that a general solution describing such an interaction consists of components associated separately with a soliton and with a perturbation thus determining the superposition of the previous ones. The effect of switching is embedded into the MSP as well as the notion of a soliton itself.

Take, for instance, the usual KdV

$$u_t + uu_x + u_{xxx} = 0, \qquad u = u(x, t). \tag{1}$$

Here a superposition formula is of the form [2]

$$u(x,t) = -3(k+\theta_x)^2 \tanh^2\left(\frac{kx - k^3t + \theta}{2}\right) + 6\theta_{xx}\tanh\left(\frac{kx - k^3t + \theta}{2}\right) + \frac{3}{2}(k+\theta_x)^2 + \frac{3}{2}\left(\frac{\theta_{xx}}{k+\theta_x}\right)^2 - 3\left(\frac{\theta_{xxx}}{k+\theta_x}\right) + \frac{3}{2}k^2, \qquad \theta = \theta(x,t), \quad k > 0 \tag{2}$$

where the function θ satisfies the equation

$$2\theta_t + 2\theta_{xxx} - \theta_x^3 - 3k\theta_x^2 - \frac{3\theta_{xx}^2}{\theta_x + k} = 0, \qquad \theta = \theta(x,t). \tag{3}$$

If we associate it with a localized perturbation in such a manner that

$$\lim_{x\to\pm\infty} \theta(x,t) = \theta_\pm = \text{const} \tag{4}$$

then before and after an interaction asymptotically far from the soliton we will have for the perturbation initially placed on the right of the last one

$$u_{\text{perturbation}_{\text{before/after}}} = \pm 6\theta_{xx} - \frac{3}{2}(k+\theta_x)^2 + \frac{3}{2}\left(\frac{\theta_{xx}}{k+\theta_x}\right)^2 - 3\left(\frac{\theta_{xxx}}{k+\theta_x}\right) + \frac{3}{2}k^2 \tag{5}$$

and respectively for the soliton itself

$$u_{\text{soliton}_{\text{before/after}}} = 3k^2\left[1 - \tanh^2\left(\frac{kx_1 - k^3t + \theta_\mp}{2}\right)\right] \tag{6}$$

(if on the left, conversely $u_{\text{perturbation}_{\text{after/before}}}$ and $u_{\text{soliton}_{\text{after/before}}}$). That is, the perturbation as well as the soliton is switched from one state to another.

As was shown in previous papers [1, 2], the approach works very well for integrable cases. Most physical models belong to the so-called nonintegrable type, however. Thus the question of whether (at least in a reasonable approximation) solitonic interactions can also be described by means of the MSP, or in other words whether they are of the same nature, is important for real research. One step in this direction is to verify the existence of an analogous switching effect for the wide class of solitonic models and arbitrary perturbations, that would indirectly indicate its universality in the above sense. Furthermore, this effect by itself can be important from the viewpoint of possible application.

With this goal in the present paper, the related experiments are carried out for several well-known equations of mathematical physics, both integrable (the above KdV) and nonintegrable (the Kawahara and regularized long wave) equations.

2. Scheme of an experiment

Consider the following experiment (see figure 1(*a*)); call it experiment 1. At some moment t_0 take a soliton (S) and perturbation (P_0) far enough from each other to avoid their mutual overlapping. Choose for simplicity a coordinate system moving with this soliton and assume for definiteness that the perturbation is initially located on the right and moves towards the previous one. Suppose further that the interaction with the soliton, say, turns over this perturbation. In figure 1 the profiles of the solution $u(x, t_i)$ and perturbation (P_0, P_1, P_2) are schematically shown for consecutive moments of time t_0, t_1, t_2. Next, consider another but

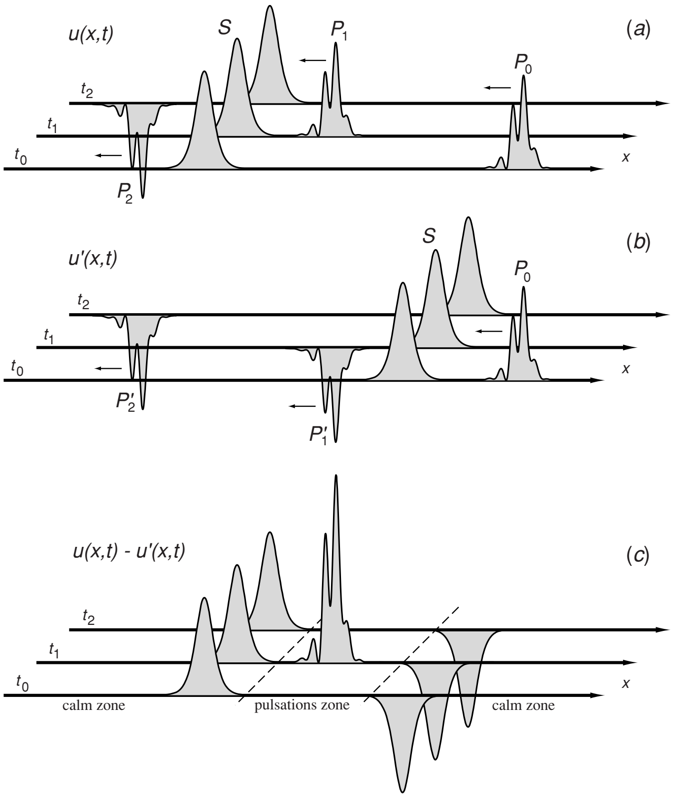

Figure 1. The principal scheme of the experiments. (*a*) Experiment 1. (*b*) Experiment 2. (*c*) The plot of the difference $u - u'$.

similar experiment, experiment 2 (figure 1(*b*)), where the soliton is shifted in comparison with the previous case. Here P_0, P_1' and P_2' again indicate the profiles of the perturbation at the same moments. Now plot the difference between the above solutions, $u(x, t_i) - u'(x, t_i)$, figure 3(*c*).

We see that between the solutions the perturbations increase while out of this region they mutually eliminate each other. The fact is that in the framework of the MSP evolution of a soliton and perturbation are independent (it comes in different dimensions), so that the result of switching can always be presented in the form [1]

$$u_{\text{perturbation}_{\text{after}}}(x, t) = S_{-\infty}^{+\infty} T(t_0, t) u_{\text{perturbation}_{\text{before}}}(x, t_0)$$

where $S_{-\infty}^{+\infty}$ is the operator determining switching only, and $T(t_0, t)$ is the operator determining evolution in the absence of a soliton for the same time (strictly speaking $t_0 \to -\infty$ and $t \to +\infty$; as an example, see (5) and (3) respectively for the KdV case). By this means the states P_2 and P_2' of our perturbation are identical, and the difference between P_1 and P_1' results from the action of the switching operator only.

Above we have considered a very conventional perturbation. In a real situation an arbitrary perturbation is collapsed to various wave packages and separate pulses moving one after another with various velocities. As a result, if we additionally introduce periodical boundary conditions, making the switching process continuous and repeated, then a distinct

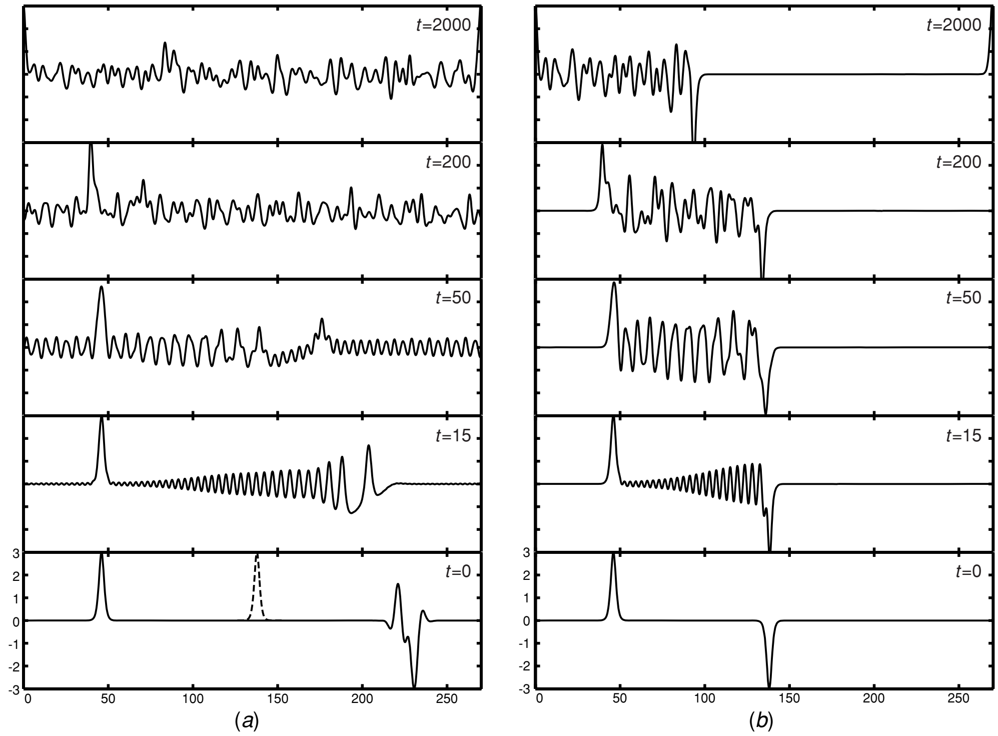

Figure 2. The experiments with the KdV equation. (*a*) Experiment 1 (the broken curve at $t = 0$ indicates the soliton position in experiment 2). (*b*) The plot of the difference $u - u'$.

picture of two different zones, a *pulsation zone* between solitons and *calm zone* outside of them, has to arise.

3. Experiments with periodical boundary conditions

First of all, consider an integrable equation, namely the KdV (1), and its soliton solution (6). Figure 2(*a*) demonstrates the typical scenario for our experiment (experiment 1) with the initially localized perturbation. As seen, with time various waves completely fill all available space. In figure 2(*b*), for the plot of the difference $u(x,t) - u'(x,t)$ the appearance of the two above-mentioned zones is clearly visible. In doing so, the amplitude of the waves in the calm zone (call it *secondary pulsations*) is about 10^{-14} at the final moment $t = 2000$ of the simulation, while the accuracy (see the comments below) in this experiment was about 10^{-16}. Some of the difference can be explained by general accumulation of errors. To have a measure of such deviations from the ideal, we will introduce the characteristic time for similar experiments t_{char}, the average time for which an initially localized perturbation spreads to all space. Here $t_{\text{char}} = 20$.

Next consider the Kawahara equation

$$u_t + uu_x - u_{xxx} + u_{xxxxx} = 0, \qquad u = u(x,t),$$

one of the known representatives of the so-called nonintegrable models. Basically, this equation is a modification of the usual KdV for situations when we have to take into account

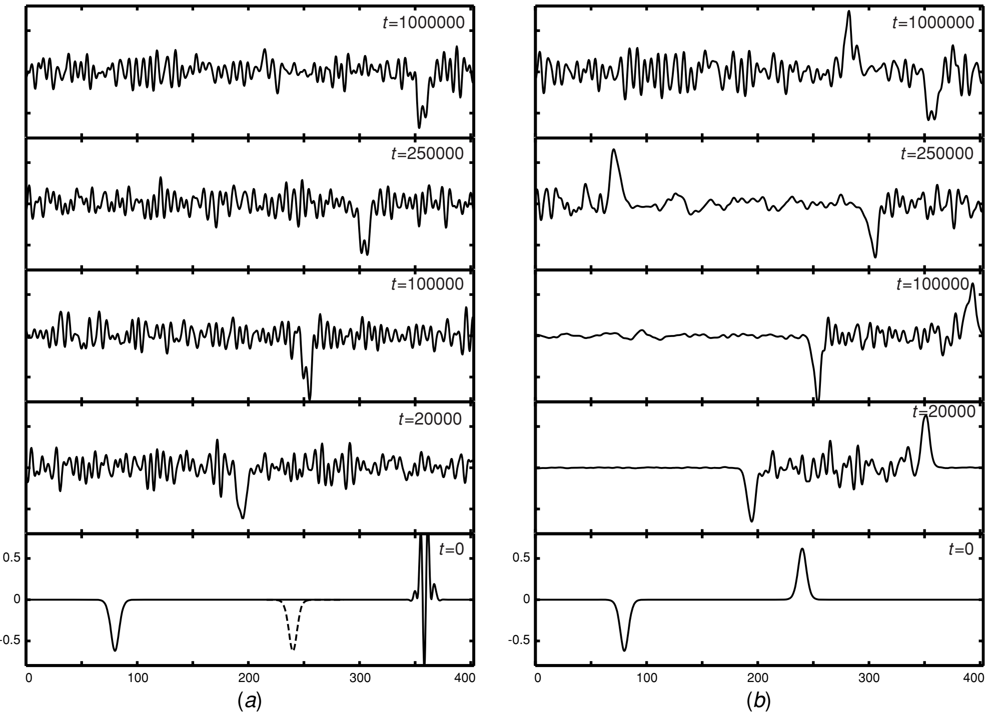

Figure 3. The experiments with the Kawahara equation. (a) Experiment 1 (the broken curve at $t = 0$ indicates the soliton position in experiment 2). (b) The plot of the difference $u - u'$.

the higher order dispersion effects to balance the nonlinearity. One of its solitonic solutions is of the form

$$-\frac{105}{169}\cosh^{-4}\left(\frac{x}{2\sqrt{13}} + \frac{18}{169\sqrt{13}}t\right)$$

(see [5] and references therein on its solutions and physical applications). Figure 3(a) demonstrates the evolution of the initial distribution in the first experiment. In the case with this equation, however, very small but perceptible secondary pulsations appear also in the calm zone. Although at times comparable with $t_{\mathrm{char}} = 100$ they cannot be taken into account, but with time they grow (for example, for the times $t = 2 \times 10^4$, 10^5, 2.5×10^5 their maximal amplitude A_{sp} is 5×10^{-3}, 3×10^{-2} and 10^{-1} respectively), and at times as great as 10^6 they are compatible with the amplitude of the waves in the pulsation zone itself (see figure 3(b)). We emphasize, however, that this takes place only after a very long time.

The next model of interest is the RLW equation

$$u_t + uu_x + u_{xxx} - u_{txx} = 0, \qquad u = u(x, t)$$

with the following solitonic solution

$$\frac{12k^2}{1 - 4k^2}\cosh^{-2}\left(kx + \frac{4k^3}{4k^2 - 1}t\right), \qquad k = \mathrm{const}.$$

In due course, this equation has attracted intended attention of researchers because of its very small inelasticity of solitonic interactions and nearness to the KdV (a review can be found in the book [6], and from previous works see, e.g., [7]).

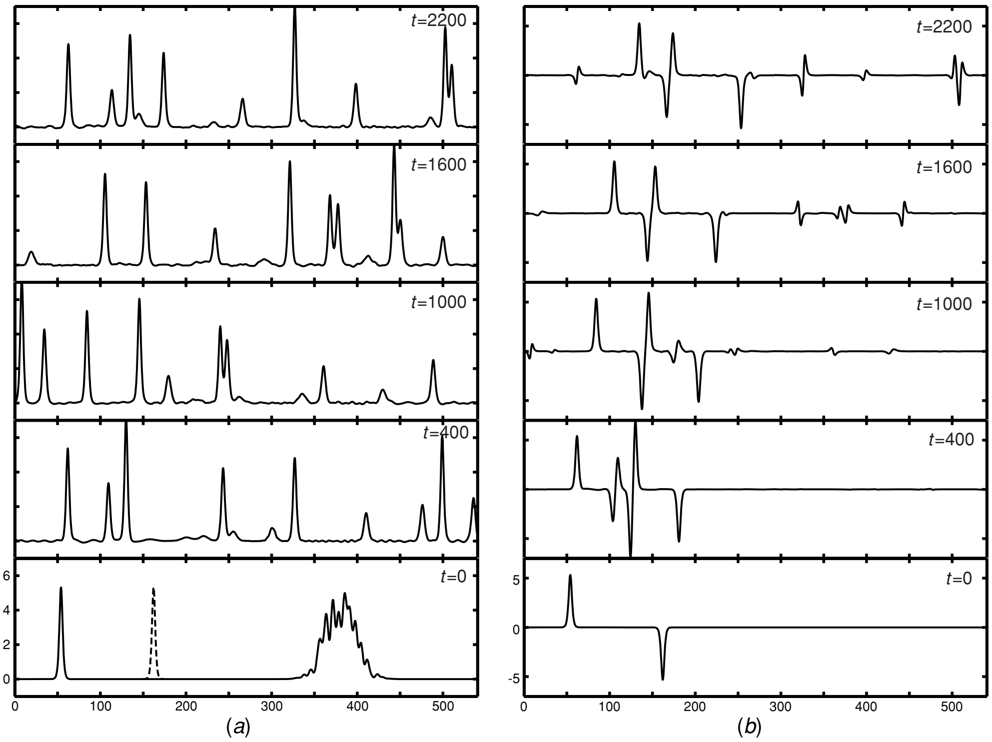

Figure 4. The experiments with the RLW equation. (*a*) Experiment 1 (the broken curve at $t = 0$ indicates the soliton position in experiment 2). (*b*) The plot of the difference $u - u'$.

In contrast to the previous equation, here an arbitrary perturbation collapses to a series of mainly separated soliton-like pulses (figure 4(*a*), experiment 1). However, the pulsation zone is easily discernible. As shown in figure 4(*b*), after some time secondary pulsations appear in the calm zone also and grow with time as well as in the previous case. Contrary to expectations (it is agreed that the RLW is close to integrable NPDEs in its properties more than any other nonintegrable model) it turned out that these secondary pulsations are more pronounced than, e.g., in the Kawahara equation case. The related dynamics is as follows: $A_{\mathrm{sp}} = 0.05, 0.3, 1.2, 2$ for $t = 400, 1000, 1600, 2200$. They became compatible with the main pulsations at times of the order of 4000 units while $t_{\mathrm{char}} = 200$.

Finally, let us say a few words about the computational aspects of the above experiments.

Firstly, remember that the periodical boundary conditions

$$u(x, t) = u(x + L, t)$$

(see the figures concerning L) were used. Secondly, the pseudospectral (Fourier expansion) technique was applied on the spatial variable with the related number of modes and time integrator for stiff problems required for the highest of them to reach an adequate accuracy for our goal. Further, whenever possible, all calculations were performed with a maximal accuracy permutable by the capacity of the processor (`long double`, 18 digits). According to the Runge principle of step-doubling, an error was $\varepsilon \sim 10^{-16}$. In difficult cases (the experiments with the Kawahara equation for very long times) an accuracy with $\varepsilon \sim 10^{-8}$ was used for a reasonable CPU time.

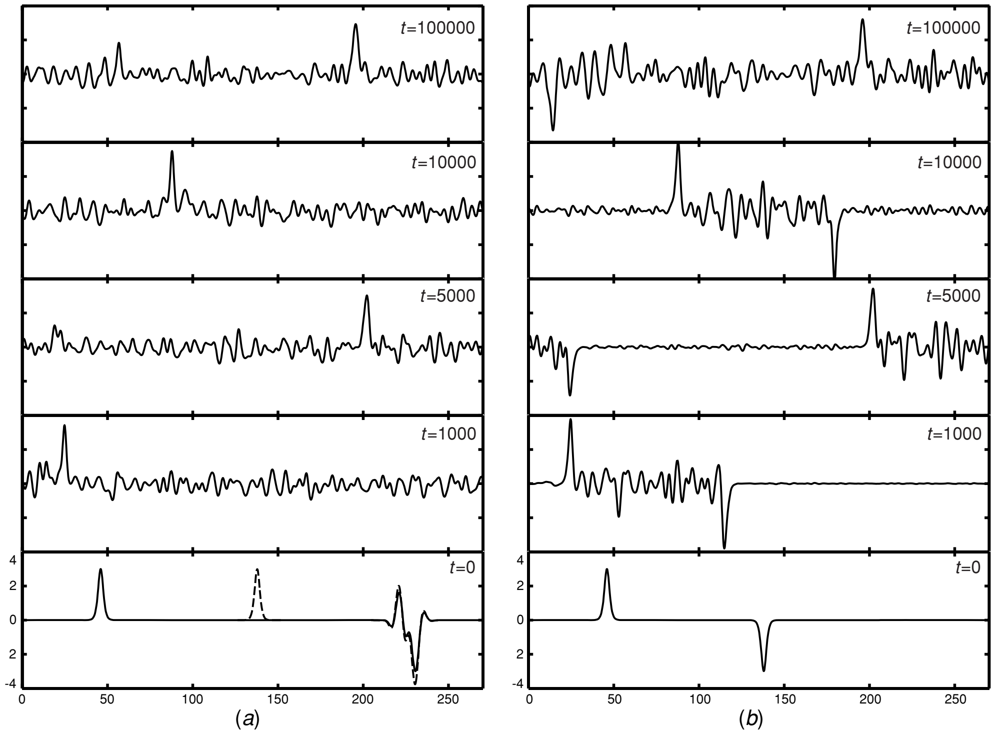

Figure 5. The experiments with the KdV equation and initial perturbations differing by a scale factor 1.001. (*a*) Experiment 1 (the broken curve at $t = 0$ schematically shows the profiles of the soliton and perturbation in experiment 2). (*b*) The plot of the difference $u - u'$.

4. Conclusion

The experiments carried out demonstrate the essential similarity between the soliton interactions in the integrable and nonintegrable models from the MSP viewpoint. It should be especially emphasized here that the secondary pulsations in the last two examples do not contradict, by themselves, the existence of exact soliton–perturbation superposition. As shown in [1] (figure 4 and the related formulae), superposition implies both elastic and inelastic interactions depending on asymptotical behaviour of a component associated with a perturbation (for instance, in (2) this is $\theta(x, t)$; recall the demand (4) for the possibility of physical separation of the soliton and a perturbation before and after a collision). In the example cited, the soliton and perturbation remain linked with each other by a small shelf even for long distances between them. As a result, the time and distance needed for full switching there are dramatically increased. For small distances, this is reflected in some defect of a perturbation envelope 'after' an interaction. In our experiments, once arisen, it will evaluate and progress even without the contributions from the following same interactions. Figures 5(*a*) and (*b*) show an experiment with the KdV where the initial perturbations insignificantly differ (by the scaling factor 1.001), that simulates, say, a defect arising after single, not full, switching. As a consequence of this, we see a picture similar to those observed for the nonintegrable models.

In conclusion, we point out one more detail in favour of this explanation. As seen from figure 3, in spite of very significant time of the simulation and innumerable collisions with other waves, there is no indication of the collapse or degradation of the soliton in the experiment with the Kawahara equation, where it would be easy to discover.

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